3 Translations of the Yoneda Lemma: An Unfaithful Functor From Math Into Something Like Stories

Kavi Duvvoori
Undergraduate, Brown University
69 Brown St. 3316, Providence, RI 02912, USA
kavi_duvvoori@brown.edu

Abstract

I present a "literary translation" of the Yoneda Lemma, a statement in categorical algebra, into a piece of prose writing, followed by some commentary. This project hopes to investigate the cultural activity of mathematics as a linguistic and aesthetic practice, and to see what happens when some of the patterns in a piece of mathematical writing are used in the seemingly quite foreign context of literary writing. A brief discussion of the motivation, influences, execution, and questions behind it follows.

1 Disclaimers

The "Second Translation" is a bit verbose and not too helpful $-$ you are invited (and encouraged) to skim or skip it. The statement of the text Yoneda Lemma is not taken from a single other account, but formulated by me from several other statements of it.

All metaphors are to be used at the readers own risk. The translator takes no responsibility for any harm $-$ social, academic, emotional, ethical, financial, physical, or otherwise $-$ caused by improper use of the over-large generalizations and unjustified associations made here.

2 Original Text

1. $C = [Obj(C), Hom : Obj(C) \times Obj(C) \to \text{Set}, \circ, Id : Obj(C) \to Hom(C, C)]$, a category
2. $F \in \text{Set}^C$
3. $A \in \text{Obj}(C) \Rightarrow \text{Hom}(A, -) \in \text{Set}^C$ where
   (a) $f \in \text{Hom}(X, Y) \mapsto \text{Hom}(A, f) = [g \in \text{Hom}(A, X) \mapsto f \circ g \in \text{Hom}(A, Y)]$
   (b) $B \mapsto \text{Hom}(A, B)$

Lemma: (Yoneda)

1. $\text{Nat}(\text{Hom}(A, -), F) \cong_I F(A)$
2. $I \in \text{Nat}(\text{Hom}(A, -), F), F(A))$ where $\text{Nat}(\text{Hom}(X, -), Y), Y(X) \in \text{Set}^{\text{Set}^C \times C}$

3 First Translation

Let C be any locally small category, i.e. a class of objects Obj and a morphism Hom function from the class of objects squared to the class of all sets, a composition operation on adjacent morphisms, and an identity function Id from the class of objects to the class of sets. Let F be any functor from C to Set, the category of sets as objects with maps as arrows. Let A be an object in C. Then, the covariant hom-functor at A is a functor from C to Set defined as follows: Hom at A of a morphism f from an object X to an object Y is taken to the map between the hom-set of A and X and the hom-set of A and Y that acts by composition with f.

The Yoneda Lemma states that the set of natural transformations between the covariant hom-functor at A and F is isomorphic with F of A, and this isomorphism is natural when both sides are considered as functors from the category of functors from C to Set cross C to Set.

4 Second Translation

Well need to know what a *category*, a *functor*, a *hom-functor* and a *natural transformation* is to state the Yoneda Lemma.

Take some category C a collection of objects and a set of arrows or morphisms between each pair of objects. C must be locally small for every pair of objects the collections of arrows forms a set, the hom-set. There are some rules to the arrangement of arrows any path of two continuous arrows can be composed into a corresponding direct path a single arrow from the source to the destination. Also, each object has an identity arrow to itself that does nothing – it leaves all other arrows unchanged when they are composed with it on either side. A category can describe, like a sort of map, the global structure of the objects in a mathematical theory, for example the category Set of all sets and functions between them, or Top of all topological spaces with continuous functions. They can also describe the structure of other mathematical things for example a partially-ordered set, where there is a single arrow from one object to another when the first object is greater than or equal to the second. Categories provide a powerful common language for mathematics, revealing underlying commonalities and recurring structures.

A functor is like a function between categories. It takes the objects and arrows in a category to a choice of corresponding objects and arrows in another category. The only rules are that it respects composition a functor of a composition of objects is the same as the composition of the functor of the objects and identities it takes identities to identities. A functor translates information about the arrangement of one category into another category. Let F be a functor from C to Set assigning a set to each object in C and a function between the corresponding sets to each arrow in C.

We can now define the covariant hom-functor at any object A in C it takes any object B to the set of arrows from A to B. For an arrow f, it gives the map between the hom-sets from A to X and from A to Y given by composition with f: for each arrow from A to Y, it gives the arrow from A to Z given by composing the arrow with f, i.e. by going along f after going on g. It is straightforward to check that this is in fact a functor from C to Set. Essentially, it describes how A fits inside C Hom at A forgets about everything except how things are connected with A.

A natural transformation from F to Hom at A is a correspondence between the images of the two functors: for any object X in C it gives an arrow n sub X between F of X and Hom at A of X, so that the arrows commute with the image of all other arrows f from X to Y in C: n sub Y composed after F of f is the same as Hom at A of f composed after n sub X you get the same path if you take the natural transformations first or last. Basically, a natural transformation is a way of associating the picture F gives of C to the picture Hom at A gives of C.

The Yoneda Lemma states that the set of natural transformations from F to the hom-functor at A is isomorphic to F of A in a natural way i.e., when both sides of this isomorphism are considered as functors taking a pair of a functor and an object in C to a corresponding set, this isomorphism is a natural transformation. The Yoneda Lemma allows for the definition of representable functors and shows that in a variety of contexts the functions into an object are the right generalization of objects. Roughly, it says that the structure of a category whose objects and arrows are associated with sets and maps, as most useful categories are, can be completely described by the structure of the maps between the representations of objects in terms of the sets of maps between them! To understand a category of math, we just need to understand the maps around each mathematical object.

5 Third Translation

First, learn cartography. No one changes road-signs. Trains leave on time. Flights follow their scheduled routes and blink their positions along monitors over your head and you try to sleep. Ships are registered with the proper authorities. Camels walk where theyre told.

There is language, call it Set. People turn words into words. Authors write encyclopedias, news reports, self-help books, and nursery rhymes which can be found in basements or the Library of Congress.

You start somewhere, C. Look around.

The things around you arrange themselves in terms of old words, remembered hallways that you can walk around the corners of when you close your eyes in the right weather. That house is green and boarded and wooden and high like your neighbors, which you walked into once and wished you hadnt. The city is made out of streets arranged in a grid like youve seen in foldout and googled maps, made out of rectangles that you learned the name and shape of with wooden blocks that fit into each other in your Montessori pre-school. You see the way branches rise as the explanation your mother gave about photosynthesis when you, loud and fragile, asked how plants grow (or maybe what she told you about tree spirits waving to meet the running sun.)

You have a childrens dictionary, or an atlas, or a bird watching book, or a Guidebook for the Budget Traveler F. It says some things.

Go somewhere. Pick something up, A. See how the world holds it. Consider what this billboard or bird or 11th story window, or cloud, or left sock sees. Call what it says A's Hom, type it up, and put it on a shelf somewhere.

Now hold the stories side by side. Look at one, and then the other. Your left sock tells you about an earthworm experimenting with air, deciding on things, and now you step a little to the left. The book prefers the history of the sidewalk, the experimentations of the early family-run concrete companies, the century old firmness of the stone marked and dated by Herbert & Co, which a foot rests on.

Write an essay comparing and contrasting (but mostly comparing) A's Hom and F. Make some bubble charts. Notice characters overlapping motivations, and influences in structure and worldview. Try out thesis statements, and support with well-cited examples your properly ordered topic sentences.

Now youre drowning in a forest of metaphors but look around (says Yoneda). Make some maps your wrists and fingers feel coastlines turning smoothly forward, then a sudden cutting back. Make islands play off of islands, trace names hovering around the bends in rivers which come out of your mouth like the way the water pumps forward, and which keep sounding right when you step in them. Consider the coastline of a chin or palm or democratic constitution. See how to get from here to there, what this says about that. The maps make matching shapes; they're the same place. Maybe as you look you know where you are, how you got here.

6 A Sketch of an Apology

As a student of mathematics as well as of literary arts, I am deeply interested in language, structure, meaning and the relations between the three: how a syntax can give rise to a semantics. The mathematical proof is a genre of writing, shaped by often unmentioned aesthetic values around clarity and elegance, while literary writings take place in linguistic systems where they often take advantage of subtle, interesting, and not un-mathematical patterns. My awareness of, and interest in, this life of mathematics was sharpened and inflamed by the remarkable and unique collection of essays on the "interplay of mathematics and narrative" collected by Apostolos Dioxadix and Barry Mazur in *Circles Disturbed*[2]. The two particular problems that preoccupy me are as to the literary structure of mathematical texts - the sequential development, rhetorical devices, and choices of notation that make a proof clear or beautiful - and about the relationships between precise and careful patterns and orthogonal components of something like randomness in poetry, prose, and combinations thereof.

I have often heard, and also used, the phrase "language of mathematics" and I wanted to make that literal: if math is a language, made and spoken by a community of specialists, can that language be translated into one or the other of the languages that we otherwise speak? One type of answer given to this question is that of popular science and mathematics - by paraphrase and metaphor: that curving space-time is a rubber-sheet stretching, or that a group is a formal way of encoding a collection of "symmetries" for example. In a sense, these are literal translations, trying to precisely recreate the semantic meaning of a piece of mathematics in a radically different context. The translation I wanted to produce - wondered if it would be possible to produce - would instead be a figurative one, that attempted in a highly personal and idiosyncratic way to convey my sense of the "shape" and "feel" of the mathematics, while entirely failing to give the reader a direct understanding of the original formal and technical statement. The end product, in its several parts, hopefully suggests the tentative, personal, and contingent nature of any translation, including the ones that textbooks and professors give.

I chose the Yoneda Lemma because of its place in category theory, the discipline known, even among mathematicians, for "abstract nonsense." The lemma is a statement that is a deep and powerful tool in subjects like algebraic geometry, but that is at an enormous remove from the more ordinary objects of even mathematicians' other thought and understanding.

Related practices to this translation include that of the homophonic and semantic translation of poetry, practiced especially by the Oulipo, and the translations between mediums Hollywood studios make each week. The hope is that by carrying a structure from the odd and very polite context of standard math into rowdy language, something a bit foreign and new can be found; the reality is maybe a bit more complex. I wonder about the reverse as well - could the shifts and catharses of a poem be translated into a series of mathematical symbols? This is left as an exercise to the reader.

References

[1] J. Adámek, H. Herrlich, and GE Strecker, *Abstract and Concrete Categories: The Joy of Cats*, 2009, Dover Publications.

[2] A. Dioxadis and B. Mazur, *Circles Disturbed: The Interplay of Mathematics and Narrative*, 2012, Princeton University Press.

[3] B. Mazur, "When is One Thing Equal to Some Other Thing", *Proofs and Other Dilemmas: Mathematics and Philosophy*, 2008, p. 221.

[4] E. Moure, *My Beloved Wager: Essays From a Writing Practice*, 2009, Newest Press.

[5] G. Perec, *Species of Spaces and Other Pieces*, 2008, Penguin Classics.

[6] ML Saunders, *Categories for the Working Mathematician*, 2nd edn. (1998), Springer.

[7] P. Tuntha-Obas, *Trespasses*, 1st ed. (2006), O Books

[8] V. Tasic, *Mathematics and the Roots of Postmodern Thought*, 2009, Oxford University Press.